Interrupted coarsening in a driven kinetically constrained Ising chain

Suzanne M. Fielding*
Department of Physics and Astronomy, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom
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We introduce a driven version of the one-dimensional kinetically constrained spin chain [J. Jackle and S. Eisinger, Zeitschr. Phys. B **84**, 115 (1991)]. In its original undriven version, this model shows anomalous coarsening following a quench to a low temperature, with an equilibration time that diverges as $\sim\exp(1/T^2)$ for $T\to 0$. We show that driving of constant rate $\dot{\gamma}$ interrupts coarsening and stabilizes the chain in a state analogous to that of a coarsening chain of age $1/\dot{\gamma}$. We present an analytical theory for this steady state, and demonstrate it to be in excellent agreement with our simulation results.

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I. INTRODUCTION

Glassy systems relax very slowly at low temperatures. They therefore remain out of equilibrium for long times and exhibit ageing [1]: the time scale for response to an external perturbation (or for the decay of correlations) increases with the ''waiting time'' t_w since the system was quenched to the low temperature, and thus eventually far exceeds the experimental time scale. Time translational invariance is lost.

As a result of this dynamical sluggishness, glassy systems are highly susceptible to steady external driving, even when the driving rate $\dot{\gamma}$ is small. (One example of $\dot{\gamma}$ is shear rate in a rheological system.) Typically, such driving interrupts ageing and restores a time-translationally invariant (steady) nonequilibrium state in which the time scale defined by the inverse driving rate plays a role analogous to the waiting time t_w of the ageing regime [2]. This scenario was first investigated in the context of neural networks [3,4] and has subsequently been reproduced in the diffusion of a particle in a random potential [5,6]; in driven mode-coupling equations of the mean-field p-spin model [7]; and in a driven version of Bouchaud's trap model [8–10]. The results of the latter two studies were separately exploited to propose a general framework for the study of ''soft glassy materials'' [11–15] in which intrinsic rheological ageing is interrupted by driving (shear straining) and loading (shear stress) [12].

In this paper, we introduce a driven version of another glassy model: the one-dimensional Ising chain with an asymmetric kinetic constraint. The original undriven model (introduced by Jäckle and Eisinger [16]) shows anomalous slow coarsening (''ageing'') following a quench to a low temperature $T\ll 1$, with an ergodic time that diverges as $\tau\sim\exp(\text{const}/T^2)$ for $T\to 0$, as solved exactly by Sollich and Evans [17]. Particularly attractive features of the model are (i) that its glassiness emerges as a direct result of the dynamical constraint (without the need for any underlying assumption of quenched disorder) and (ii) that it contains explicit spatial interactions while being simple enough to allow analytical progress. In what follows, our central result will be that steady driving interrupts coarsening and stabilizes the chain at an apparent age $O(1/\dot{\gamma})$, consistently with the scenario described above.

Ageing and driven glassy systems in general violate the equilibrium fluctuation dissipation theorem (FDT) [18]. Remarkably, however, in the ageing limit $t_w\to\infty$ of many mean-field [19,20] and some non-mean-field [21–24] models, a nontrivial modified FDT emerges and defines a nonequilibrium temperature T_{eff} [25]. Cugliandolo *et al.* [25] proposed that an equivalent temperature should apply in the driven limit $\dot{\gamma}\to 0$, and that $T_{\text{eff}}(\dot{\gamma}\to 0)$ and $T_{\text{eff}}(t_w\to\infty)$ should coincide. To date, however, the evidence [7,26] for this is rather limited. An added motivation for the present paper, therefore, is that the model defined here can be used (in future work [27]) to test this scenario further via comparison of its FDT predictions with those of the undriven model [28].

The paper is structured as follows. In Sec. II we summarize the results of Ref. [17] for the coarsening dynamics of the undriven chain. We then define the novel driving rules in Sec. III. We present simulation and analytical results for the steadily driven chain in Secs. IV and V respectively, demonstrating that driving of rate $\dot{\gamma}$ halts coarsening at an effective age $1/\dot{\gamma}$. We summarize and give an outlook for future work in Sec. VI.

II. COARSENING IN THE UNDRIVEN MODEL

The model consists of a chain of L Ising spins $s_i\in 0,1$ $(1<i\leqslant L)$ in a uniform field of unit magnitude, which is oriented such that a spin's energy $E_i(s_i)=s_i$. Periodic boundary conditions apply: the left neighbor of s_1 is s_L. The dynamics are subject to the following constraint: at any time, only those spins whose left neighbor is up are allowed to flip. For these ''mobile'' spins, the rate of down flips is 1 while the rate of up flips is $\epsilon=\exp(-1/T)$. The *equilibrium* distribution is unaffected by this constraint: detailed balance is obeyed and the static distribution is the trivial one prescribed by the Hamiltonian $H=\Sigma_{i=1}^{L}s_i$. In contrast, the dynamics are rendered very slow at low temperatures, for which the equilibrium concentration of up spins (that facilitate the dynamics) $c=\epsilon/(1+\epsilon)$ is small.

In this section we review Ref. [17], which solved exactly

*Present address: Department of Physics and Astronomy & Polymer IRC, University of Leeds, Leeds LS2 9JT, United Kingdom.

the relaxation of a chain prepared at a low temperature $T \ll 1$ via rapid quench at time $t_w=0$ from a high initial temperature $T_i=\infty$. At any time t_w during the relaxation, the authors of Ref. [17] described the system's state using the concept of ''domains.'' As shown by the vertical lines in

$$\ldots 1|\cdot\cdot 1|\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot 1|\cdot 1|\cdot\cdot\cdot\cdot 1|\cdot\cdot\cdot 1|1|1|\cdot 1\ldots$$

(in which the 0 spins are represented by ''$\cdot$'' for clarity) a domain is defined as an up spin, and all the down spins separating it from the next up spin to the left.

Immediately after the quench, the average domain length $\bar{d} \equiv 1/c = O(1)$. The system then relaxes towards the low temperature's equilibrium state in which $\bar{d}=O(1/\epsilon)$, and thus in which, in the limit $\epsilon \to 0$, there is zero probability of finding an up spin in any finite length of chain. Hence, in this limit ($\epsilon \to 0$ at fixed chain length) the down flipping of spins is irreversible, and the relaxation comprises a coarsening process in which adjacent domains progressively coalesce with one another.

The mechanism for this coalescence is as follows. Consider, at some time after quench, a domain of length $d \ll 1/\epsilon$ together with its left-bounding up spin. For the purposes of the present argument, we assume that the left-bounding spin is ''clamped'' and consider how the right-bounding up spin relaxes. Because of the constraint, before the relaxation can occur a facilitating up spin has to be generated immediately to the left of this spin, via a propagation of the up state rightwards from the left-bounding up spin. The relaxation is thus impeded by an energy barrier, the height of which is the maximal number of spins that are ever up within the (original) domain at any instant during this relaxation process. A central result of Ref. [17] is that for domain lengths $2^{n-1} < d \leqslant 2^n$ this barrier scales as n, leading to a relaxation time scale $O(\epsilon^{-n})$.

Hence in the limit $\epsilon \to 0$ the dynamics comprise well separated stages, the n^{th} of which has time scale ϵ^{-n} and results in domains of index n being destroyed by coalescence with their right neighbors. In logarithmic time, $\nu_w = -\ln t_w / \ln \epsilon = T \ln(t_w)$, the n^{th} stage collapses to the point $\nu_w = n$. The average domain size $\bar{d}$ thus exhibits stepwise increases at successive integer values of ν_w, as seen in Fig. 1 of Ref. [17].

Within this coarsening regime ($\bar{d} \ll 1/\epsilon$), the full domain length distribution $P(d)$ can be calculated using an exact independent interval treatment [29], which states that no correlations can build up in the length of adjacent domains provided none are present in the initial state. At stage n the distribution obeys [17]

$$\partial_\tau P(d,\tau) = \sum_{2^{n-1} < d' \leqslant 2^n} P(d-d',\tau)[-\partial_\tau P(d',\tau)] \quad (1)$$

in which the rescaled time $\tau = t\epsilon^n$ can take any positive value $\tau > 0$ in the limit $\epsilon \to 0$. Equation (1) describes the coalescence of the ''active'' domains of length $d' \leqslant 2^n$ with neighboring domains of length $d-d'$. Its initial condition is the domain length distribution at the end of stage $n-1$ of the dynamics, denoted $P_n(d) = P(d, \tau \to 0)$. Using generating functions it can be shown that

$$P_{n+1}(d) = P_n(d) + \sum_{d'=1}^{d-1} P_n(d-d') P_n^{\text{act}}(d') - \frac{1}{2} \sum_{d'=1}^{d-1} P_n^{\text{act}}(d-d') P_n^{\text{act}}(d') + \cdots \quad (2)$$

in which $P_n^{\text{act}}(d)$ is the active part of the distribution (zero for $d > 2^n$) and in which $\ldots$ denotes a series of convolutions of increasing order. Equation (2) holds only for $d > 2^n$: all active domains ($d \leqslant 2^n$) disappear in the nth stage.

The weight of the distribution shifts to larger d at each stage of coarsening. See Fig. 2 of Ref. [17]. A scaling limit is approached for large stage number n: the rescaled distribution $\tilde{P}_n(x = d/2^{n-1}) = 2^{n-1} P_n(d)$ converges to the limit $\tilde{P}_{\text{age}}(x)$ which obeys, for $x > 2$, the scaling counterpart of Eq. (2),

$$\frac{1}{2}\tilde{P}_{\text{age}}\left(\frac{x}{2}\right) = \tilde{P}_{\text{age}}(x) + \int_0^x dx' \tilde{P}_{\text{age}}(x-x') \tilde{P}_{\text{age}}^{\text{act}}(x') - \frac{1}{2}\int_0^x dx' \tilde{P}_{\text{age}}^{\text{act}}(x-x') \tilde{P}_{\text{age}}^{\text{act}}(x') + \cdots . \quad (3)$$

In Ref. [17] the exact solution for $\tilde{P}_{\text{age}}(x)$ was shown [via the re-summed Laplace transform of Eq. (3)] to be

$$\tilde{P}_{\text{age}}(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} \int_1^\infty \prod_{r=1}^{m} \frac{dx_r}{x_r} \delta\left(\sum_{s=1}^{m} x_s - x \right). \quad (4)$$

III. DEFINITION OF THE DRIVEN MODEL

In this section we incorporate non-Hamiltonian driving into the model. As a preliminary step, though, we redefine the *relaxational* dynamics slightly, extending state space such that each spin $s_i \in -1,0,1$ and redefining the Hamiltonian $H = \Sigma_{i=1}^L |s_i|$. The uniform field has thus been replaced by a potential well for each spin, with a minimum at $s_i = 0$. As before the dynamics are constrained: only those spins for which the left neighbor has a value 1 or -1 are allowed to flip via the usual thermal processes in which the transition rate for $s_i : 1 \to 0$ and for $s_i : -1 \to 0$ is 1, and for $s_i : 0 \to 1$ and $s_i : 0 \to -1$ is ϵ.

So far, of course, the model can be exactly mapped onto the original one by a trivial relabeling $s_i = -1 \to s_i = 1$ and rescaling $\epsilon \to \epsilon/2$. Our motivation for introducing the -1 state is to make a loose analogy with glassy rheological models [10,30,31] in which a local state of high energy (here $|s_i| = 1$) can have either positive or negative local stress (here $|s_i| = +1, \quad |s_i| = -1$). If we define a global stress $\sigma = (1/L)\Sigma_{i=1}^L s_i$, the -1 state allows a state of *macroscopically* zero stress, which still has internal local stresses (some positive, some negative) and a nonzero rate of internal dynamical rearrangements.

We now incorporate steady driving into this three state

version. Loosely this mimics, in a stochastic way, the standard rheological experiment of applying shear strain of constant rate $\dot{\gamma}$. To do this, we impose a flip rate of $\dot{\gamma}$ for s_i: $-1\to 0$ and $\dot{\gamma}$ for s_i:$0\to 1$. This is *additional* to the constrained rates defined above and *free of the kinetic constraint*. The driving rates ω for this extended model can therefore be summarized as follows:

$$\omega(s_i:0\to 1)=|s_{i-1}|\epsilon+\dot{\gamma},$$

$$\omega(s_i:0\to -1)=|s_{i-1}|\epsilon,$$

$$\omega(s_i:1\to 0)=|s_{i-1}|,$$

$$\omega(s_i:-1\to 0)=|s_{i-1}|+\dot{\gamma}, \tag{5}$$

in which periodic boundary conditions impose $s_0=s_L$.

This stochastic straining clearly tends to increase the global stress, as required intuitively. We note, though, that our stochastic rules only make sense for $\dot{\gamma}\geqslant 0$. For negative $\dot{\gamma}$ we would redefine the driven contribution to the rates as equal to $|\dot{\gamma}|$ for the transition s_i:$1\to 0$ and $|\dot{\gamma}|$ for s_i:$0\to -1$. The model is in this sense singular at $\dot{\gamma}=0$.

IV. SIMULATION RESULTS FOR THE STEADILY SHEARED MODEL

We simulated the driven chain using a waiting time Monte Carlo technique combined with a binary search algorithm for locating the mobile spins, following Ref. [17]. For each run we initialized the chain either in equilibrium (with $\dot{\gamma}=0$) at a low temperature $T=-1/\log\epsilon\ll 1$, or by quenching to T from $T=\infty$. For the quenched case, we then let the system relax according to the undriven rules ($\dot{\gamma}=0$) until a start-up time t_s, when we set $\dot{\gamma}$ to the nonzero, constant value of interest. Equilibrium initialization formally corresponds to a quenched chain subsequently allowed to relax until $t_s=\infty$, and in this case we applied the nonzero $\dot{\gamma}$ from the start of the simulation. In order to explore the hypothesis that driving restores a steady state analogous to the state of a coarsening chain of age $t_w=1/\dot{\gamma}$, we chose values of $1/\dot{\gamma}$ corresponding to the waiting times studied in Ref. [17] for the undriven chain. Specifically, therefore, we are interested in the low temperature limit in which $\epsilon\to 0$, $\dot{\gamma}\to 0$, at fixed values of $\nu=\ln(\dot{\gamma})/\ln(\epsilon)=T\ln(1/\dot{\gamma})$ that are large compared to 1 (weak driving), but small enough that the system remains far from equilibrium. (To avoid possible confusion we note that ν becomes large as $\dot{\gamma}$ becomes *small*, since $\dot{\gamma}=\epsilon^{\nu}$ with $\epsilon\ll 1$. Although this is at first sight a counter intuitive way to characterize the shear rate, we chose this particular definition for ν as the closest possible analogy to that of ν_w for the undriven chain.)

In each run we monitored the stress σ and total energy E as functions of time t_w. Results for the quenched initial condition with start-up time $t_s=0$ are shown in Fig. 1, for $\epsilon=0.01$ and various values of ν. As ν becomes larger ($\dot{\gamma}$ smaller) we can make the following observations. At early times, E and σ have time evolutions that are independent of $\dot{\gamma}$, and that (we have checked) are the same as those of an undriven chain. In contrast, after a crossover at time $O(1/\dot{\gamma})$, E, and σ approach steady-state values. We have checked by repeating the simulation for the different initial conditions described above that these steady values do not depend upon the initial state or (for the quenched case) the start-up time t_s. For the remainder of the paper, we shall be concerned only with the ultimate steady state, and not the kinetics of its formation.

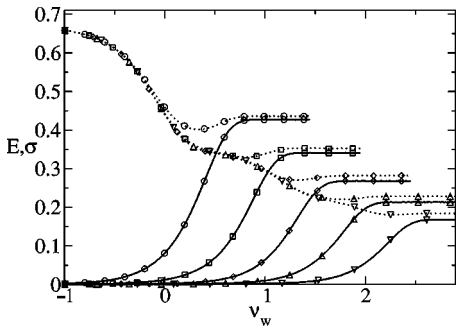


FIG. 1. Stress σ (solid lines) and energy E (dashed lines) vs scaled time $\nu_w=-\ln(t_w)/\ln(\epsilon)$ following a quench at time $t_w=0$, with driving commenced also at $t_s=0$; for all curves $\epsilon=0.01$. The parameter $\nu\equiv\ln\dot{\gamma}/\ln\epsilon$ has values 0.5 (○), 1.0 (□), 1.5 (◇), 2.0 (△), and 2.5 (▽). Each curve was obtained from a single run for a chain of length $L=2^{16}$.

The steady-state stress is replotted in Fig. 2 as a function of $-\nu$ (which increases with $\dot{\gamma}$) for various small values of $\epsilon\ll 1$. [In rheological parlance, $\sigma(\dot{\gamma})$ is the flow curve.] It appears to be approaching a steplike function as temperature is tracked towards zero, comprising plateaux separated by jumpwise discontinuities at integer values of ν.

For all the steady states studied, we found the concentration of -1 spins to be small [$O(\epsilon)$], and hence that (to within such corrections) the stress, the energy, and the concentration of $+1$ spins coincide (consistently with the results

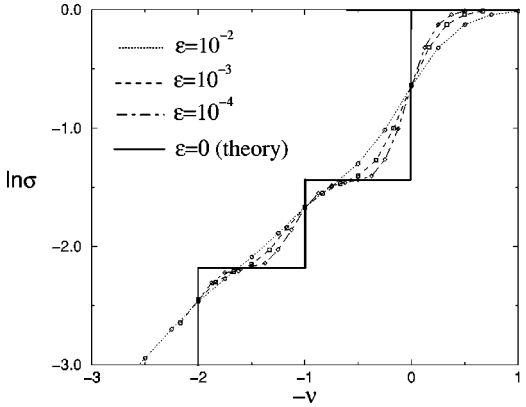


FIG. 2. Steady-state flow curves plotted on a log scale vs $-\nu=-\ln\dot{\gamma}/\ln\epsilon$. Simulation results for three values of $\epsilon=\exp(1/T)$ are shown, each obtained from a single run for a spin chain of length $L=2^{16}$. Bold line: theoretical prediction for $T\to 0$.

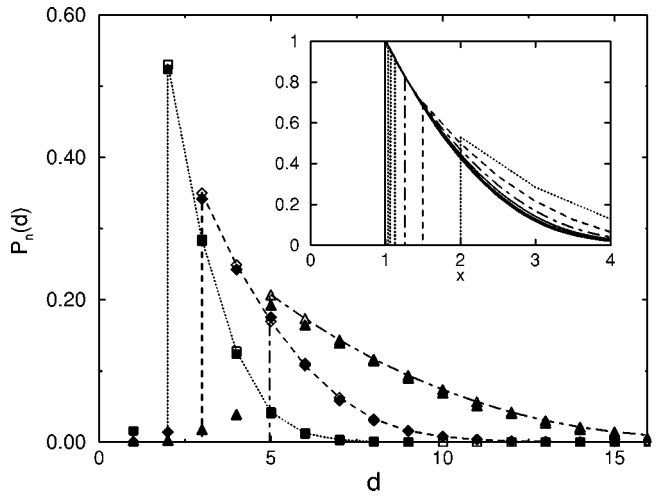


FIG. 3. Domain length distributions $P_n(d)$ on the plateau $\nu = n-1/2$ of the flow curves. Open symbols and lines: theoretical results for $n=1$ (squares), 2 (diamonds), 3 (triangles). Full symbols: simulation results for a chain of length $L=2^{16}$ and $\epsilon = 10^{-4}$ $(n=1,2)$, and $\epsilon=10^{-3}$ $(n=3)$. Inset: scaled predictions $2^{n-1}P_n(d=2^{n-1}x)$ vs x for $n=1,\ldots,16$.

of Fig. 1). Using the usual domain description (in which for definiteness we neglect the -1 spins, taking only 1 spins to constitute the domain boundaries), we have then $\sigma = 1/\overline{d}$ where $\overline{d}$ is the average domain length.

We note the striking similarity between the dependence of $\ln \overline{d}$ upon scaled waiting time $\nu_w = T \ln t_w$ in the coarsening chain (Fig. 1 of Ref. [17]), and the dependence of $\ln \sigma = -\ln \overline{d}$ upon the scaled driving rate $-\nu = -T \ln(1/\dot{\gamma})$ in the steadily driven chain (Fig. 2 of this paper). This already gives us a strong indication that the steady state of a chain driven at rate $\dot{\gamma}$ is analogous to the state of a coarsening chain of age $t_w = 1/\dot{\gamma}$.

This hypothesis is confirmed by our simulation data for the full domain length distribution $P(d)$ on the developing plateaux of the flow curves. In particular, we find that the distribution for the steadily driven chain at a given ν is closely analogous to that of a coarsening chain for $\nu_w = \nu$. For a given ν (or ν_w), both display a discontinuity at the same ν (ν_w) dependent cutoff and have very similar averages. Both shift abruptly to larger values of d as ν (ν_w) crosses successive integers, but are unchanged as ν (ν_w) is swept between integers. In order to maintain the closest notational analogy with Ref. [17], we denote by $P_n(d)$ the distribution $P(d)$ in the limit of small ϵ for values of ν such that $n-1<\nu<n$. Our results for $P_n(d)$ for $\nu = 0.5, 1.5, 2.5$, corresponding to $n=1,2,3$, are shown in Fig. 3 and are, as just noted, very similar to the counterpart results of Fig. 2 of Ref. [17] for the coarsening chain.

In the coarsening chain, the discontinuous shift of $P(d)$ as ν_w crosses successive integers arose from the waiting time crossing the time scales for successive coarsening stages. Likewise in the driven chain it arises from the inverse driving rate crossing these same time scales. (See the theory in Sec. V for more details.)

These simulation results therefore show that (consistently with the phenomenology of other driven glassy models) driving interrupts coarsening and stabilizes the chain in a state that is strikingly analogous to that of a coarsening chain of apparent age $O(1/\dot{\gamma})$.

V. THEORY

We now present a theory aimed at calculating the steady-state distribution $P_n(d)$ just defined. We assume from the outset that no correlations exist between the lengths of adjacent domains. We also assume (consistently with our above remarks) that the concentration of -1 spins is negligible. We will return to justify these assumptions in more detail at the end of this section.

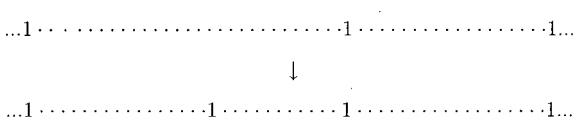
Consider, then, a chain with an $O(1)$ population of $+1$ spins and a complementary $O(1)$ population of 0 spins. In the absence of driving, the only process affecting the chain would be the coarsening described above, with down flipping of 1 spins occurring irreversibly (for fixed chain length and $\epsilon \to 0$), and the average domain length increasing as domains of length $2^{n-1} < d \leqslant 2^n$ are destroyed on a time scale $O(\epsilon^{-n})$ by coalescence with their right neighbors.

In the steadily driven chain, this coarsening process is balanced by intradomain driven up flipping of spins $s_i: 0 \to 1$. For the finite domain lengths to which we shall restrict ourselves, this driven domain intersection occurs on a time scale $O(1/\dot{\gamma}) \equiv O(\epsilon^{-\nu})$. For a noninteger value of ν such that $n-1<\nu<n$, this time scale sits between the time-scales $\epsilon^{-(n-1)}$ and ϵ^{-n} for the adjacent coarsening stages $n-1$ and n (recall Sec. II), and separates from them in the limit $\epsilon \to 0$.

On this driving time scale, therefore, any domain of index $n' \leqslant n-1$ (present either as a remnant of the initial condition, or as a result of a driven up flip $0 \to 1$ within an existing domain of length $d > 2^{n-1}$ a distance $d \leqslant 2^{n-1}$ from the domain boundary) must relax infinitely quickly. We thus expect $\lim_{\epsilon \to 0} P_n(d) = 0$ for such domain lengths $d \leqslant 2^{n-1} \equiv d_c$. This defines the cutoff length d_c observed in the simulation data above.

Domains labeled by $n' > n-1$, i.e., of length $d > d_c$, on the other hand, coarsen infinitely slowly on the driving time scale: the only dynamical processes that can affect these "long" domains are those initiated by driven up flip of an intradomain 0 spin. This up flip can occur at a distance that is either $\leqslant d_c$ or $> d_c$ from either end, giving four separate cases:

(a) If the original long domain was longer than $2d_c+1$, and if the up flip occurred at a distance of at least d_c+1 from both ends, we see creation of two shorter domains that are still both "long" in the sense that $d > d_c$,



(For definiteness all the diagrams in this section assume an arbitrary cutoff value $d_c = 4$.)

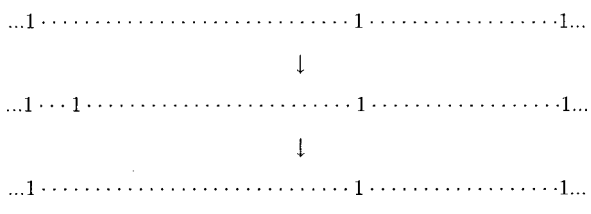
Considering the chain as a whole, such processes lead to the destruction of domains of length $d>2d_c+1$ at a rate

$$\dot{\gamma}(d-2d_c-1)P_n(d) \tag{6}$$

and to the creation of domains of length $d>d_c$ at a rate

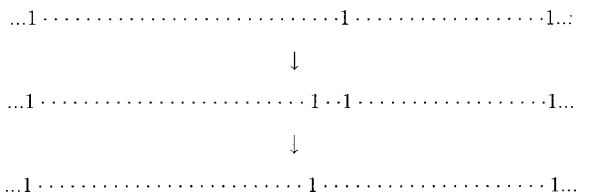
$$2\dot{\gamma}\sum_{d'=d+d_c+1}^{\infty} P_n(d'). \tag{7}$$

(b) If the up flip occurred within a distance d_c of the left-hand end of the original domain, but at a distance greater than d_c+1 from the right-hand end we see a process such as



which does not need to be considered further since it results in no net change.

(c) If the up flip occurred within a distance d_c of the right-hand end of the original domain, but at a distance greater than d_c from the left-hand end, the ''short'' (right) subdomain will relax immediately by coalescence with its right neighbor,



leaving two ''long'' ($d>d_c$) domains, with the boundary between them shifted to the left. Processes such as this lead to the creation of domains of length d at a rate

$$\dot{\gamma}\sum_{d'=d+1}^{d+d_c} P_n(d')+\dot{\gamma}\sum_{d'=1}^{d_c}\left[\sum_{d''=d_c+d'+1}^{\infty} P_n(d'')\right]P_n(d-d') \tag{8}$$

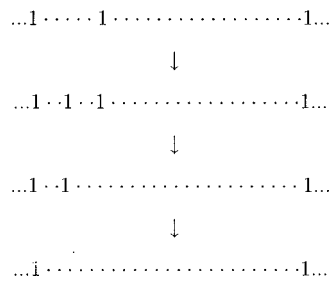
and to the destruction of domains of length d at a rate

$$\dot{\gamma}\Theta(d-2d_c)d_cP_n(d)+\dot{\gamma}\Theta(2d_c-d+1)(d-d_c-1)\times P_n(d)+\dot{\gamma}P_n(d)\left[\sum_{d'=d_c+1}^{2d_c}(d'-d_c-1)\times P_n(d')+d_c\sum_{d'=2d_c+1}^{\infty}P_n(d')\right],$$

where the discrete Theta function is defined by

$$\Theta(n-m)=1 \quad \text{for} \quad n>m$$
$$=0 \quad \text{for} \quad n\leqslant m. \tag{9}$$

(d) If the original domain was of length $d\leqslant 2d_c$, both the subdomains could be short ($d\leqslant d_c$). For this class as a whole (i.e., averaging over the position of the up-flipped spin) there are two subclasses of possible outcome, which by symmetry each occurs with a probability one-half. First, the freshly flipped up spin could relax before the right-bounding spin of the original domain and we would see no net change. Alternatively, we could see



which is essentially ''aided coalescence.'' For the chain as a whole, processes such as these lead to destruction of domains of length d at a rate

$$\dot{\gamma}P_n(d)\frac{1}{2}(2d_c-d+1)+\frac{1}{2}P_n(d)\dot{\gamma}\times\sum_{d'=d_c+1}^{2d_c}P_n(d')(2d_c-d'+1)$$

and creation at a rate

$$\frac{\dot{\gamma}}{2}\sum_{d'=d_c+1}^{2d_c}(2d_c-d'+1)P_n(d')P_n(d-d'). \tag{10}$$

Combining all these processes we get an evolution equation $\partial_t P_n(d)=\ldots$ for the ''long'' domains $d>d_c$, which we set equal to zero (steady state) and solve numerically using an iterative procedure. The solutions for $n=1,2,3$ for which $d_c=1,2,4$, respectively are marked as open symbols in Fig. 3 and give excellent agreement with the simulation results. We also used these solutions to calculate the stress $\sigma=1/\overline{d}$. As expected, this exhibits discontinuous jumps at integer values of ν as the driving time scale crosses successive coarsening time scales and $P(d)$ shifts discontinuously to larger d. It is marked as the solid line in Fig. 2 and again agrees excellently with the simulation data.

As $d_c\to\infty$, a scaling limit $d_cP(d)=\tilde{P}(x=d/d_c)$ is approached. See the inset of Fig. 3 and Fig. 4. Taking the limit $d_c\to\infty$ at fixed $x=d/d_c$ in the steady-state equation just derived, we find that this scaling state must obey the equation

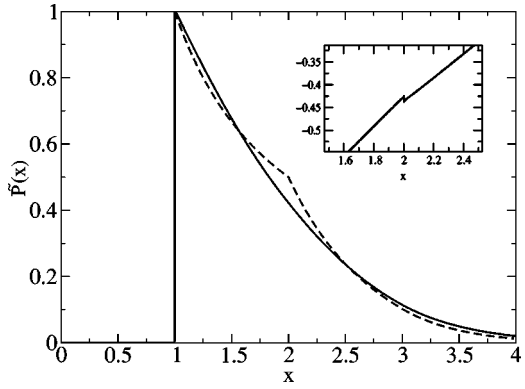


FIG. 4. Scaling distribution $\tilde{P}(x)$ for the driven chain (solid line) and $\tilde{P}_{\rm age}(x)$ for the coarsening chain (dashed line). Inset: derivative $d\tilde{P}/dx$ for the driven distribution, showing a slight discontinuity (too small to be discernible in the undifferentiated data of the main figure) at $x=2$.

$$0=-g(x)\tilde{P}(x)+2\int_x^\infty dx'\tilde{P}(x')-\int_x^{x+1}dx'\tilde{P}(x') + \int_0^2 dx' f(x')\tilde{P}(x-x'). \tag{11}$$

We note for use below that the function $f(x)$ (thus defined) is discontinuous at $x=1$, and the first derivative $g^{(1)}(x)$ of g is likewise discontinuous at $x=2$. In principle, Eq. (11) contains all the information needed to calculate $\tilde{P}$ analytically. For the counterpart state $\tilde{P}_{\rm age}(x)$ in the undriven chain, the closed expression [Eq. (4)] was found [17] as the self-consistent solution of a simple algebraic relation between the Laplace transforms G and H of $\tilde{P}_{\rm age}(x)$ and $\tilde{P}_{\rm age}(x)\Theta(2-x)$ respectively. The corresponding transform of Eq. (11) for the driven case is a complicated differential relation between G, H, $G^{(1)}$, and $H^{(1)}$, and we have been unable find a self-consistent analytic solution. However our numerical results (see Fig. 4) demonstrate that $\tilde{P}(x)$ is (as expected) very similar to its undriven counterpart $\tilde{P}_{\rm age}(x)$: both have a unit Heaviside discontinuity at $x=1$ and show similar decay for $x>1$. Although the discontinuity in the first derivative $\tilde{P}^{(1)}_{\rm age}(x)$ at $x=2$ (strongly apparent in the dashed curve of Fig. 4) is less noticeable in the solid curve for the driven state $\tilde{P}$, it is revealed by numerical differentiation in the inset of Fig. 4.

It was shown in Ref. [17] that the ageing scaling distribution $\tilde{P}_{\rm age}$ has a finite discontinuity in its kth derivative at $x=k+1$ for all integer $k\geqslant 0$. We have already seen numerically that the driven scaling distribution shares the discontinuities for $k=0,1$. We shall now outline an analytical argument that can be used to show that in fact the driven distribution shares *all* of these discontinuities. We confine ourselves to $k\geqslant 1$ since our analysis has already captured the discontinuity in $\tilde{P}$ itself at $x=1$. Differentiating Eq. (11) once we get

$$-g(x)\tilde{P}^{(1)}(x)=g^{(1)}(x)\tilde{P}(x)+\tilde{P}(x)+\tilde{P}(1+x) - \int_0^2 dx' f(x')\tilde{P}^{(1)}(x-x'). \tag{12}$$

On the right-hand side (RHS) of this expression, $g^{(1)}(x)$ is discontinuous at $x=2$ (as noted above), while the integral over the infinite discontinuity $\delta(x-x'-1)$ of the differential $\tilde{P}^{(1)}(x-x')$ in the last term's integrand gives $\Theta(3-x)f(x-1)$, which is discontinuous at $x=2$. All other terms on the RHS are continuous for all $x>1$. Hence $P^{(1)}(x)$ has a finite discontinuity at $x=2$. By performing successive differentiations, we can extend this argument to arbitrarily high k: $\tilde{P}^{(k)}(x)$ has a finite discontinuity at $x=k+1$ for all $k\geqslant 0$.

Our central result can therefore be summarized as follows. We have shown that a steadily driven chain approaches a scaling state in the limit of small $\dot{\gamma}$. We have shown that this scaling state is strikingly analogous to the counterpart scaling state of a coarsening chain of age $t_{\rm w}=1/\dot{\gamma}$. In particular, both states have a unit Heaviside discontinuity at $x=1$, and (more generally) a finite discontinuity in $\tilde{P}^{(k)}(x)$ at $x=k+1$.

Despite the striking qualitative similarities in the distributions $\tilde{P}(x)$ and $\tilde{P}_{\rm age}(x)$, there are obvious quantitative discrepancies. This is not inconsistent with the phenomenology of other glassy models: the ageing state of Bouchaud's trap model [8], for example, is analogous but not identical to its steadily driven counterpart [10]. It is nonetheless instructive to consider the origin of the discrepancies. An obvious candidate is our introduction in the driven model of the -1 spin state. However, in steady state the population of such spins is small. Indeed, we have checked that domain distributions produced by driven simulations without the -1 state agree with those that include the -1 state to within $O(\epsilon)$.

The discrepancies must therefore be of dynamical origin. During coarsening, the only process is domain coalescence. Therefore, information can only propagate up the distribution (to larger d). In this way the scaling limit $\tilde{P}_{\rm age}(x)$ is completely determined by $P(x')$ for $x'<x$, as seen in Eq. (3). In contrast, in the driven chain $\tilde{P}(x)$ is connected to all $x'\to\infty$, since driven domain intersection acts on domains of all lengths; see Eq. (11). Furthermore, in the ageing chain the domain lengths set the dynamical rates only via the coarsening time scales ϵ^{-n}. In contrast, the driving dynamics are further sensitive to domain length through the additional feature that a longer domain is more likely to be intersected.

The most apparent difference between $\tilde{P}_{\rm age}(x)$ and $\tilde{P}(x)$ is the much smaller discontinuity in $\tilde{P}^{(1)}(x)$ at $x=2$. This can be explained as follows. The discontinuity in $\tilde{P}^{(1)}(x)$ arises from an interplay between the discontinuity in $g^{(1)}(x)$ at $x=2$ and in $f(x)$ at $x=1$ [recall Eq. (12)]. Physically, $g(x)$ is the rate at which domains of scaling length x are destroyed. This is discontinuous because driven intersection of a domain of scaled length $x<2$ can produce two active domains (scaling length <1), whereas only one can be active for $x>2$. Similarly, the discontinuous part of $f(x')$ re-

sults from the production of domains of length x via aided coalescence (process d above) which can only occur for $x<2$. These two effects are of comparable (not identical) magnitude but opposite sign resulting in only a small discontinuity in $\tilde{P}^{(1)}(x)$. In contrast, the discontinuity in $\tilde{P}^{(1)}_{\rm age}(x)$ arises from a single process, the origin of which can be seen from Eq. (3). The second term on the RHS describes the production of domains of length x via coalescence of an active domain of length x' with a right neighbor of length $x-x'$. However if $x-x'$ is itself active, the right neighbor can relax before the x' domain, resulting in a domain of length $>x$. The rate of producing domains of length x is thus reduced, as encoded in third term on the RHS. The cutoff in the active distribution at $x=2$ means that the derivative of this term is discontinuous at $x=4$. When transferred to the next coarsening stage [encoded in the left-hand side of the equation], this discontinuity appears in $\tilde{P}^{(1)}_{\rm age}(2)$. The higher convolutions not shown in Eq. (3) are smooth enough not to affect this argument.

We finally return to justify our two assumptions: first, that the population of -1 spins is negligible in the steady state. As noted above, we have already numerically observed this population to be $O(\epsilon)$ (small); we are now in a position to show this theoretically, as follows. Over a time scale $\lesssim 1/\dot{\gamma}$, the only mechanism in which -1 spins can be created involves propagation of the facilitating state (-1 or 1), via constrained thermal activation, to within a distance $d<d_c$ to the right of one of the existing facilitating spins. Denoting by $\tau_c(d)$ the time scale on which domains of length d coarsen, the time scale upon which a -1 spin is created at a distance d from an existing facilitator is $\tau_c(d)/\epsilon$. Once created, such a spin will relax back to the 0 state on a time scale $\tau_c(d)$. Hence in the steady state, the population of -1 states must be $O(\epsilon)$, which is indeed small in the limit considered here.

Our second assumption was that of ''independent intervals.'' This was used in Ref. [17] for the coarsening process of the undriven chain, and in that case is provably exact. We have not been able to prove its strict validity for this driven case. However, our simulation results show that the relevant correlation function $C=\Sigma_{j=0}^{n}(d_j-\langle d\rangle)(d_{j+1}-\langle d\rangle)/\Sigma_{j=0}^{n}(d_j-\langle d\rangle)^2$ (in which d_j is the length of jth domain from the left-hand end of the chain, and n is the total number of domains) is not greater than 10^{-3} in any of the steady states considered; the assumption, therefore, is likely to be reasonable. Note that our numerical observation of a nonzero correlation function could still be consistent with the independent interval approximation being exact in the limit $\epsilon\to 0$ at fixed ν. Indeed, we have observed that, as ϵ is tracked downwards, the numerical value of the correlation function gets smaller.

VI. SUMMARY AND OUTLOOK

We have incorporated driving dynamics into the kinetically constrained spin chain of Ref. [16], and presented simulation results showing that the coarsening dynamics of the undriven chain (as solved in Ref. [17]) are interrupted by steady driving. Consistent with the broader glassy literature [2–7,9,10] we have found that steady driving stabilizes the chain at an apparent age set by the inverse driving rate. We have presented a theory for this steady state, demonstrating it to be in excellent agreement with our simulation results. We have shown that a scaling state is approached at small $\dot{\gamma}$, and that this scaling state has very similar properties to its counterpart scaling state reached at long times in the coarsening chain.

We now outlook some possible directions for future work. Above, we focused on a chain that is steadily driven at a constant rate $\dot{\gamma}$. We also noted the loose analogy of this driving scenario to that of constant shear rate in a rheological system. In the spirit of this rheological connection, we can identify possible analogues of two other standard rheological tests—step strain and step stress—which it would be interesting to investigate further. For a step strain of size γ_0 we promote, at the time t_w of strain application, a fraction γ_0 of -1 spins (chosen randomly) according to $s_i:-1\to 0$ and of 0 spins (again chosen randomly) $s_i:0\to 1$. We do this *without regard to the kinetic constraint*. For all other times the system merely relaxes under its undriven constrained dynamics. (Note that the step at t_w is just the ''impulsive limit'' of the above steady shear case: $\dot{\gamma}dt=\gamma_0$, with $dt\to 0$ and $\dot{\gamma}\to\infty$.) For a step stress of size σ_0, we apply the same dynamics just defined for the step strain up until the time t_w^+. (We can merely rename γ_0 by σ_0 because the ''spring constant'' $k\equiv 1$.) For $t>t_w$ we implement the ''constant'' strain-rate dynamics defined above, but with $\dot{\gamma}$ continuously adjusted to ensure that σ_0 remains (on average) a constant.

As noted in the introduction, it would also be interesting to study FDT in the driven steady state (at constant $\dot{\gamma}$) to see if any effective temperature emerges, and (if so) whether it coincides with any effective FDT temperature of a coarsening chain of age $t_w=1/\dot{\gamma}$. This is the subject of a forthcoming publication [27].

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